MATRIX LYAPUNOV INEQUALITIES FOR ORDINARY AND ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

ANTONIO CAÑADA AND SALVADOR VILLEGAS

Abstract. This paper is devoted to the study of L_p Lyapunov-type inequalities for linear systems of equations with Neumann boundary conditions and for any constant $p \geq 1$. We consider ordinary and elliptic problems. The results obtained in the linear case are combined with Schauder fixed point theorem to provide new results about the existence and uniqueness of solutions for resonant nonlinear problems. The proof uses in a fundamental way the nontrivial relation between the best Lyapunov constants and the minimum value of some especial minimization problems.

1. Introduction

Let us consider the linear Neumann boundary problem

$$u''(x) + a(x)u(x) = 0,\ x \in (0, L),\ u'(0) = u'(L) = 0 \tag{1.1}$$

and let $1 \leq p \leq \infty$ be given. If function a satisfies

$$a \in L^p(0, L) \setminus \{0\},\ \int_0^L a(x)\ dx \geq 0, \tag{1.2}$$

L_p-Lyapunov inequality provides optimal necessary conditions for boundary value problem (1.1) to have nontrivial solutions, given in terms of the L^p norm, $\|\cdot\|_p$, of the function a^+, where $a^+(x) = \max\{a(x), 0\}$(see [11] and [12] for the case $p = 1$ and [4], [27] for the case $1 < p \leq \infty$).

In particular, under the restriction (1.2) for $p = 1$, L_1-Lyapunov inequality may be used to prove that (1.1) has only the trivial solution if function a satisfies

$$\int_0^L a^+(x)\ dx \leq 4/L \tag{1.3}$$

In a similar way, under (1.2) for $p = \infty$, $L_\infty-$Lyapunov inequality may be used to prove that (1.1) has only the trivial solution if function a satisfies

$$a^+ \prec \pi^2/L^2, \tag{1.4}$$

2000 *Mathematics Subject Classification.* 34B15, 34B05, 35J20, 35J25, 35J65.

Key words and phrases. Neumann boundary value problems, Matrix Lyapunov inequalities, ordinary differential equations, elliptic partial differential equations, resonant problems.

The authors have been supported by the Ministry of Education and Science of Spain (MTM2005-01331).

where for $c, d \in L^1(0,L)$, we write $c \prec d$ if $c(x) \leq d(x)$ for a.e. $x \in [0,L]$ and $c(x) < d(x)$ on a set of positive measure. Moreover, (1.3) and (1.4) are, respectively, optimal L_1 and L_∞ restrictions (see Remark 2 below).

If $p = \infty$, assumptions (1.2) and (1.4) are a nonuniform nonresonance condition with respect to the two first eigenvalues $\lambda_0 = 0$ and $\lambda_1 = \pi^2/L^2$ of the eigenvalue problem

$$u''(x) + \lambda u(x) = 0, \; x \in (0,L), \; u'(0) = u'(L) = 0 \tag{1.5}$$

(see [21]) while if $p = 1$, (1.3) was first introduced by Lyapunov under Dirichlet boundary conditions (see [11], chapter XI, for some generalizations and historic references and [7] for L_1-Lyapunov inequality at higher eigenvalues).

It is clear that (1.3) and (1.4) are not related. A natural link between them arises if L_p-Lyapunov inequalities, for $1 < p < \infty$, are considered and then one examines what happens if $p \to 1^+$ and $p \to \infty$ ([4]). One of the main applications of Lyapunov inequalities is its use in the study of nonlinear resonant problems.

Different authors have generalized the $L_\infty-$Lyapunov inequality (1.2)-(1.4) to vector differential equations of the form

$$u''(x) + A(x)u(x) = 0, \; x \in (0,L) \tag{1.6}$$

where $A(\cdot)$ is a real and continuous $n \times n$ symmetric matrix valued function, together with different boundary conditions. These L_∞ generalizations have been given not only at the two first eigenvalues but also at higher eigenvalues of (1.5) and they have been used in the study of resonant nonlinear problems ([1], [3], [14], [17], [26]). Also, some abstract versions for semilinear equations in Hilbert spaces and applications to elliptic problems and semilinear wave equations have been given in [2], [10], [19] and [20]. In spite of its interest in the study of different questions such as stability theory, the calculation of lower bounds on eigenvalue problems, etc. ([9], [11], [27]), the use of $L_\infty-$Lyapunov inequalities in the study of nonlinear resonant problems only allows a weak interaction between the nonlinear term and the spectrum of the linear part. For example, using the $L_\infty-$Lyapunov inequalities showed in [14] for the periodic boundary value problem (see also [1] and [3]), it may be proved that if there exist real symmetric matrices P and Q with eigenvalues $p_1 \leq \cdots \leq p_n$ and $q_1 \leq \cdots \leq q_n$, respectively, such that

$$P \leq G''(u) \leq Q, \; \forall \, u \in \mathbb{R}^n \tag{1.7}$$

and such that

$$\bigcup_{i=1}^{n} [p_i, q_i] \cap \{k^2 : k \in \mathbb{N} \cup \{0\}\} = \emptyset, \tag{1.8}$$

then, for each continuous and $2\pi-$periodic function h, the periodic problem

$$u''(x)+G'(u(x)) = h(x), x \in (0,2\pi), u(0)-u(2\pi) = u'(0)-u'(2\pi) = 0, \tag{1.9}$$

has a unique solution. Here $G : \mathbb{R}^n \to \mathbb{R}$ is a C^2-mapping and the relation $C \le D$ between $n \times n$ matrices means that $D - C$ is positive semi-definite. Now, by using the variational characterization of the eigenvalues of a real symmetric matrix, it may be easily deduced that (1.7) and (1.8) imply that the eigenvalues $g_1(u) \le \cdots \le g_n(u)$ of the matrix $G''(u)$, satisfy

$$p_i \le g_i(u) \le q_i, \ \forall \ u \in \mathbb{R}^n. \tag{1.10}$$

Consequently each continuous function $g_i(u), \ 1 \le i \le n$, must fulfil

$$g_i(\mathbb{R}^n) \cap \{k^2 : k \in \mathbb{N} \cup \{0\}\} = \emptyset. \tag{1.11}$$

To the best of our knowledge, we do not know any previous work on L_p Lyapunov inequalities when $1 \le p < \infty$ for systems of the type (1.6) under Neumann boundary conditions. Really, if the restrictions on the matrix $A(x)$ are of L_p type, with $1 \le p < \infty$, it seems difficult to use the ideas contained in the mentioned papers to get new results on problems at resonance.

In the second section of this paper we provide for each p, with $1 \le p \le \infty$, optimal necessary conditions for boundary value problem

$$u''(x) + A(x)u(x) = 0, \ x \in (0, L), \ u'(0) = u'(L) = 0, \tag{1.12}$$

to have nontrivial solutions. These conditions are given in terms of the L^p norm of appropriate functions $b_{ii}(x), \ 1 \le i \le n$, related to $A(x)$ through the inequality $A(x) \le B(x), \ \forall \ x \in [0, L]$, where $B(x)$ is a diagonal matrix with entries given by $b_{ii}(x), \ 1 \le i \le n$. In particular, we can use different L_{p_i} criteria for each $1 \le i \le n$ and this confers a great generality on our results. Even in the case $p_i = \infty, \ 1 \le i \le n$, our method of proof is different from those given in previous works. In fact, we begin Section 2 with a lemma inspired from [14] and [17], where the authors studied the periodic problem. The proof that we give for this lemma suggest the way for the case when $1 \le p < \infty$, where we use in a fundamental way some previous results which have been proved in [4] and [5]. They relate, for ordinary and elliptic problems, the best Lyapunov constants to the minimum value of some especial minimization problems. If $1 < p < \infty$, this minimum value plays the same role as, respectively, the constants $4/L$ (if $p = 1$) and π^2/L^2 (if $p = \infty$) in (1.3) and (1.4) (see Lemma 2.2 below).

It is clear from the proofs given here for Neumann problem, that one can deal with other situations such as Dirichlet, periodic or mixed boundary conditions (see [6] for scalar equations). Systems like (1.6) have been considered also in [8] and [9], where the matrix $A(x)$ is not necessarily symmetric and with boundary conditions either of Dirichlet type or of antiperiodic type. The authors establish sufficient conditions for the positivity of the corresponding lower eigenvalue. These conditions involve L_1 restrictions on the spectral radius of some appropriate matrices which are calculated by using the matrix $A(x)$. It is easy to check that, even in the scalar case, these conditions are independent from classical L_1-Lyapunov inequality (1.3) and therefore, for the ordinary case, they are also independent from our results in this paper. Also, in a series of papers, W. T. Reid ([23], [24], [25]) made

an extension of (1.3) for the Dirichlet problem, but he always considered $p = 1$ (see Remark 5 below).

In Section 3 we deal with elliptic systems of the form

$$\Delta u(x) + A(x)u(x) = 0,\ x \in \Omega,\ \frac{\partial u(x)}{\partial n} = 0,\ x \in \partial\Omega \tag{1.13}$$

where Ω is a bounded and regular domain in $\mathbb{R}^N$ and $\dfrac{\partial}{\partial n}$ is the outer normal derivative on $\partial\Omega$. Here the relation between p and the dimension N may be important (see Lemma 3.1). To our knowledge, there are no previous work on L_p-Lyapunov inequalities for elliptic systems if $p \neq \infty$ (see [2] and [13], section 5, for the case $p = \infty$). Finally, we show some applications to nonlinear resonant problems. In particular, and for Neumann boundary conditions, we obtain a generalization for systems of equations of the main result given in [22] where the author treated the scalar case and where they use in the proof the duality method of Clarke and Ekeland (see Theorem 3.4 below).

2. Ordinary boundary value problems

This section will be concerned with boundary value problems of the form (1.12). We begin with a preliminar lemma on $L_\infty-$Lyapunov inequalities for (1.12), inspired from [14] and [17], where the authors studied periodic boundary conditions. Our proof suggests the way to obtain optimal L_p-Lyapunov inequalities for system (1.12) in the case $1 \leq p < \infty$.

Lemma 2.1. *Let $A(\cdot)$ be a real $n \times n$ symmetric matrix valued function with elements defined and continuous on $[0, L]$. Suppose there exist diagonal matrix functions $P(x)$ and $Q(x)$ with continuous respective entries $\delta_{kk}(x),\ 1 \leq k \leq n$ and $\mu_{kk}(x),\ 1 \leq k \leq n$, and eigenvalues $\lambda_{p(k)},\ 1 \leq k \leq n$, of the eigenvalue problem (1.5) such that*

$$P(x) \leq A(x) \leq Q(x),\ \forall\ x \in [0, L] \tag{2.1}$$

and

$$\lambda_{p(k)} < \delta_{kk}(x) \leq \mu_{kk}(x) < \lambda_{p(k)+1},\ \forall\ x \in [0, L],\ 1 \leq k \leq n. \tag{2.2}$$

Then there exists no nontrivial solution of (1.12).

Proof. Let us denote by $H^1(0, L)$ the usual Sobolev space. If $u = (u_1, \cdots, u_n) \in (H^1(0, L))^n$, is a nontrivial solution of (1.12), then

$$\int_0^L < u'(x), v'(x) > \ dx = \int_0^L < A(x)u(x), v(x) >,\ dx\ \forall\ v \in (H^1(0, L))^n, \tag{2.3}$$

where $< \cdot, \cdot >$ is the usual scalar product in $\mathbb{R}^n$. The eigenvalues of (1.5) are given by $\lambda_j = \frac{j^2\pi^2}{L^2}$, where j is an arbitrary nonnegative integer number. If φ_j is the corresponding eigenfunction to λ_j, let us introduce the space $H = H_1 \times \cdots \times H_k \times \cdots \times H_n$, where for each $1 \leq k \leq n$, H_k is the

span of the eigenfunctions $\varphi_0, \varphi_1, \cdots, \varphi_{p(k)}$. It is trivial that we can choose $\psi = (\psi_1, \cdots, \psi_n) \in H$ satisfying

$$u_k + \psi_k \in H_k^{\perp}, \ 1 \le k \le n. \tag{2.4}$$

In fact

$$\psi_k = \sum_{m=0}^{p(k)} c_m^k \varphi_m, \quad c_m^k = -\frac{\int_0^L u_k(x)\varphi_m(x)\ dx}{\int_0^L \varphi_m^2(x)\ dx}, \ 1 \le k \le n, \ 0 \le m \le p(k). \tag{2.5}$$

The main ideas to get a contradiction with the fact that u is a nontrivial solution of (1.12) are the following two inequalities. The first one is a consequence of the variational characterization of the eigenvalues of (1.5). The second one is a trivial consequence of the definition of the subspace H_k.

$$\int_0^L ((u_k + \psi_k)'(x))^2 \ dx \ge \lambda_{p(k)+1} \int_0^L ((u_k + \psi_k)(x))^2 \ dx, \ 1 \le k \le n, \tag{2.6}$$

$$\int_0^L (\psi_k)'(x))^2 \ dx \le \lambda_{p(k)} \int_0^L ((\psi_k)(x))^2 \ dx, \ 1 \le k \le n.$$

Now, from (2.3) we have

$$\int_0^L < (u+\psi)'(x), (u+\psi)'(x) > \ dx = \int_0^L < A(x)(u+\psi)(x), (u+\psi)(x) > \ dx + \tag{2.7}$$

$$\int_0^L < \psi'(x), \psi'(x) > \ dx - \int_0^L < A(x)\psi(x), \psi(x) > \ dx$$

By using (2.1) and (2.2) we deduce

$$\int_0^L < \psi'(x), \psi'(x) > \ dx - \int_0^L < A(x)\psi(x), \psi(x) > \ dx \le$$

$$\int_0^L < \psi'(x), \psi'(x) > \ dx - \int_0^L < P(x)\psi(x), \psi(x) > \ dx$$

$$= \sum_{k=1}^n \int_0^L [(\psi_k'(x))^2 - \delta_{kk}(x)(\psi_k(x))^2] \ dx \le \sum_{k=1}^n \int_0^L (\lambda_{p(k)} - \delta_{kk}(x))(\psi_k(x))^2 \ dx \le 0.$$

Consequently,

$$\int_0^L < (u+\psi)'(x), (u+\psi)'(x) > \ dx \le \int_0^L < A(x)(u+\psi)(x), (u+\psi)(x) > \ dx. \tag{2.8}$$

Also, from (2.1), (2.4), (2.6) and (2.8) we obtain
(2.9)

$$\sum_{k=1}^{n} \int_0^L \lambda_{p(k)+1} (u_k + \psi_k)^2(x)\ dx \le \sum_{k=1}^{n} \int_0^L (u_k + \psi_k)'^2(x)\ dx =$$

$$\int_0^L < (u+\psi)'(x), (u+\psi)'(x) > \ dx \le \int_0^L < A(x)(u+\psi)(x), (u+\psi)(x) > \ dx \le$$

$$\int_0^L < Q(x)(u+\psi)(x), (u+\psi)(x) > \ dx = \sum_{k=1}^{n} \int_0^L \mu_{kk}(x)(u_k + \psi_k)^2(x)\ dx$$

It follows, again from (2.2), that

$$u + \psi \equiv 0. \tag{2.10}$$

But if $u + \psi \equiv 0$, then $u = \phi = (\phi_1, \cdots, \phi_n)$ for some nontrivial $\phi \in H$. Therefore,

$$\sum_{k=1}^{n} \int_0^L \lambda_{p(k)} (\phi_k)^2(x)\ dx \ge \sum_{k=1}^{n} \int_0^L (\phi_k)'^2(x)\ dx =$$

$$\int_0^L < \phi'(x), \phi'(x)\ dx = \int_0^L < A(x)\phi(x), \phi(x) > \ dx \ge \tag{2.11}$$

$$\int_0^L < P(x)\phi(x), \phi(x) > \ dx = \sum_{k=1}^{n} \int_0^L \delta_{kk}(x)(\phi_k)^2(x)\ dx$$

Now, (2.2) implies that $u_k = \phi_k \equiv 0,\ 1 \le k \le n$, which is a contradiction with the fact that u is nontrivial. □

Remark 1. It is clear from the previous proof that if the matrix functions $P(x)$ and $Q(x)$ are constant functions P and Q, then it is not necessary to assume that they are, in addition, diagonal matrices. In fact, to carry out the proof, it is sufficient to assume that they are symmetric matrices and such that if $\delta_k,\ 1 \le k \le n$ and $\mu_k,\ 1 \le k \le n$ denote the eigenvalues of P and Q respectively, then

$$\lambda_{p(k)} < \delta_k \le \mu_k < \lambda_{p(k)+1},\ 1 \le k \le n. \tag{2.12}$$

We collect now some results which have been proved in [4], section 2. Really, if we are treating with Lyapunov inequalities for scalar ordinary problems and $1 \le p < \infty$, the constant β_p defined in the next lemma, plays the same role as $\beta_\infty = \lambda_1$, in the $L_\infty-$Lyapunov inequality (1.2)-(1.4).

Lemma 2.2. *([4]) If $1 \leq p \leq \infty$ is a given number, let us define the set X_p and the functional I_p as*

$$X_1 = \{v \in H^1(0,L) : \max_{x\in[0,L]} v(x) + \min_{x\in[0,L]} v(x) = 0\},$$

$$I_1 : X_1 \setminus \{0\} \to \mathbb{R},\ I_1(v) = \frac{\int_0^L v'^2}{\|v\|_\infty^2},$$

$$X_p = \left\{ v \in H^1(0,L) : \int_0^L |v|^{\frac{2}{p-1}} v = 0 \right\},\ \text{if}\ \ 1 < p < \infty,$$

$$I_p : X_p \setminus \{0\} \to \mathbb{R}, I_p(v) = \frac{\int_0^L v'^2}{\left(\int_0^L |v|^{\frac{2p}{p-1}} \right)^{\frac{p-1}{p}}},\ \text{if}\ \ 1 < p < \infty, \tag{2.13}$$

$$X_\infty = \{v \in H^1(0,L) :\ \int_0^L v = 0\},$$

$$I_\infty : X_\infty \setminus \{0\} \to \mathbb{R},\ I_\infty(v) = \frac{\int_0^L v'^2}{\int_0^L v^2}$$

If

$$\beta_p \equiv \min_{X_p \setminus \{0\}} \ I_p,\ 1 \leq p \leq \infty, \tag{2.14}$$

and for some $p \in [1, \infty]$, function a satisfies (1.2) and $\|a^+\|_p < \beta_p$, then (1.1) has only the trivial solution.

Remark 2. It is possible to obtain an explicit expression for β_p, as a function of p and L (see [4]). In particular, $\beta_1 = 4/L,\ \beta_\infty = \pi^2/L^2$ and β_1 is attained in a function $v \in X_1 \setminus \{0\}$ if and only there exists a nonzero constant c such that $v(x) = c(x - \frac{L}{2}), \forall x \in [0,L]$. Finally and in relation to Lyapunov inequalities, the constant β_p is optimal in the following sense (see [4]): if

$$\Sigma_p = \{a \in L^p(0,L) \backslash \{0\} : \int_0^L a(x)\,dx \geq 0 \text{ and (1.1) has nontrivial solutions } \}$$

then

$$\beta_1 \equiv \inf_{a \in \Sigma_1} \ \|a^+\|_1,\ \beta_p \equiv \min_{a \in \Sigma_p} \ \|a^+\|_p,\ 1 < p \leq \infty.$$

We return to system (1.12). From now on, we assume that the matrix function $A(\cdot) \in \Lambda$ where Λ is defined as

[Λ] *The set of real $n \times n$ symmetric matrix valued function $A(\cdot)$, with continuous element functions $a_{ij}(x),\ 1 \leq i,j \leq n,\ x \in [0,L]$, such that (1.12) has not nontrivial constant solutions and*

$$\int_0^L < A(x)k, k > dx \geq 0,\ \forall\ k \in \mathbb{R}^n.$$

The main result of this section is the following.

Theorem 2.3. *Let $A(\cdot) \in \Lambda$ be such that there exist a diagonal matrix $B(x)$ with continuous entries $b_{ii}(x)$, and $p_i \in [1,\infty],\ 1 \leq i \leq n$, satisfying*

$$\begin{gathered} A(x) \leq B(x),\ \forall\ x \in [0,L], \\ \|b_{ii}^+\|_{p_i} < \beta_{p_i},\ \text{if } p_i \in (1,\infty], \quad \|b_{ii}^+\|_{p_i} \leq \beta_{p_i},\ \text{if } p_i = 1. \end{gathered} \tag{2.15}$$

Then, there exists no nontrivial solution of the vector boundary value problem (1.12).

Proof. If $u \in (H^1(0,L))^n$ is any nontrivial solution of (1.12), we have

$$\int_0^L < u'(x), v'(x) >= \int_0^L < A(x)u(x), v(x) >,\ \forall\ v \in (H^1(0,L))^n.$$

In particular, we have

$$\begin{gathered} \int_0^L < u'(x), u'(x) >= \int_0^L < A(x)u(x), u(x) >, \\ \int_0^L < A(x)u(x), k >= \int_0^L < A(x)k, u(x) >= 0,\ \forall\ k \in \mathbb{R}^n \end{gathered} \tag{2.16}$$

Therefore, for each $k \in \mathbb{R}^n$, we have

$$\int_0^L < (u(x)+k)', (u(x)+k)' >= \int_0^L < u'(x), u'(x) >=$$

$$\int_0^L < A(x)u(x), u(x) > \leq \int_0^L < A(x)u(x), u(x) > + \int_0^L < A(x)u(x), k > +$$

$$\int_0^L < A(x)k, u(x) > + \int_0^L < A(x)k, k >=$$

$$\int_0^L < A(x)(u(x)+k), u(x)+k > \leq \int_0^L < B(x)(u(x)+k), u(x)+k > .$$

If $u = (u_i)$, then for each $i, 1 \leq i \leq n$, we choose $k_i \in \mathbb{R}$ satisfying $u_i + k_i \in X_{p_i}$, the set defined in Lemma 2.2. By using previous inequality, Lemma

2.2 and Hölder inequality, we obtain

$$\begin{gathered}\sum_{i=1}^{n}\beta_{p_i}\|(u_i+k_i)^2\|_{\frac{p_i}{p_i-1}}\le\sum_{i=1}^{n}\int_0^L(u_i(x)+k_i)'^2\le\\ \sum_{i=1}^{n}\int_0^L b_{ii}^{+}(x)(u_i(x)+k_i)^2\le\sum_{i=1}^{n}\|b_{ii}^{+}\|_{p_i}\|(u_i+k_i)^2\|_{\frac{p_i}{p_i-1}},\end{gathered}\tag{2.17}$$

where

$$\tfrac{p_i}{p_i-1}=\infty,\quad \text{if } p_i=1$$

$$\tfrac{p_i}{p_i-1}=1,\quad \text{if } p_i=\infty.$$

Therefore from (2.15) we have

$$\sum_{i=1}^{n}(\beta_{p_i}-\|b_{ii}^{+}\|_{p_i})\|(u_i+k_i)^2\|_{\frac{p_i}{p_i-1}}\le 0.\tag{2.18}$$

On the other hand, since u is a nontrivial function, $u+k$ is also a nontrivial function. Indeed, if $u+k$ is identically zero, we deduce that (1.12) has the nontrivial and constant solution $-k$ which is a contradiction with the hypothesis $A(\cdot)\in\Lambda$.

Now, if $u+k$ is nontrivial, some component, say, u_j+k_j is nontrivial. If $p_j\in(1,\infty]$, then $(\beta_{p_j}-\|b_{jj}^{+}\|_{p_j})\|(u_j+k_j)^2\|_{\frac{p_j}{p_j-1}}$ is strictly positive and from (2.15), all the other summands in (2.18) are nonnegative. This is a contradiction.

If $p_j=1$, since β_1 is only attained in nontrivial functions of the form $v(x)=c(x-\frac{L}{2})$, and $v'(0)\neq 0$, we have

$$\beta_{p_j}\|(u_j+k_j)^2\|_{\frac{p_j}{p_j-1}}<\int_0^L(u_j(x)+k_j)'^2.$$

Then (2.17) and (2.18) are both strict inequalities and this is again a contradiction. □

Remark 3. Previous Theorem is optimal in the following sense. For any given positive numbers $\gamma_i,\ 1\le i\le n$, such that at least one of them, say γ_j, satisfies

$$\gamma_j>\beta_{p_j},\ \text{for some } p_j\in[1,\infty],\tag{2.19}$$

there exists a diagonal $n\times n$ matrix $A(\cdot)\in\Lambda$ with continuous entries $a_{ii}(x),\ 1\le i\le n$, satisfying $\|a_{ii}^{+}\|_{p_i}<\gamma_i,\ 1\le i\le n$ and such that the boundary value problem (1.12) has nontrivial solutions. To see this, if γ_j satisfies (2.19), then there exists some continuous function $a(x)$, not identically zero, with $\int_0^L a(x)\ dx\ge 0$, and $\|a^{+}\|_{p_j}<\gamma_j$, such that the scalar problem

$$w''(x)+a(x)w(x)=0,\ x\in(0,L),\quad w'(0)=w'(L)=0,$$

has nontrivial solutions (see the remark after Lemma 2.2). Then, to get our purpose, it is sufficient to take $a_{jj}(x) = a(x)$ and $a_{ii}(x) = \delta \in \mathbb{R}^+$, if $i \neq j$, with δ sufficiently small.

As an application of Theorem 2.3 we have the following corollary.

Corollary 2.4. *Let $A(\cdot) \in \Lambda$ and, for each $x \in [0, L]$, let us denote by $\rho(x)$ the spectral radius of the matrix $A(x)$. If the function $\rho(\cdot)$ satisfies one of the following conditions:*

(1) $\|\rho^+\|_1 \leq \beta_1$,
(2) *There is some $p \in (1, \infty]$ such that $\|\rho^+\|_p < \beta_p$,*

Then there exists no nontrivial solution of (1.12).

Proof. It is trivial, taking into account the previous Theorem and the inequality

$$A(x) \leq \rho(x) I_n, \ \forall \ x \in [0, L], \tag{2.20}$$

where I_n is the $n \times n$ identity matrix. □

Remark 4. The authors introduced in [14] and [17] similar conditions for periodic problems and $p_i = \infty$, $1 \leq i \leq n$. Our method of proof, where we strongly use the minimization problems considered in Lemma 2.2, does possible the consideration of the cases $p \in [1, \infty)$, which to the best of our knowledge are new. In particular, if $p \in [1, \infty)$, the function $\rho(x)$ may cross an arbitrary number of eigenvalues of the problem (1.5). Also, by using our methods one can deal with other boundary conditions and more general second order equations (see, for the scalar case, Remark 5 in [4] and Theorem 2.1 in [6]).

Remark 5. In this remark we show some relations between previous Corollary and some results contained in [23], [24] and [25] for Dirichlet boundary conditions.

If $A(\cdot)$ satisfies

[H] $A(x)$, $x \in [0, L]$ *is a continuous and positive semi-definite matrix function such that* $\det A(x) \neq 0$ *for some* $x \in [0, L]$

(here $\det A(x)$ means the determinant of the matrix $A(x)$) and

$$\int_0^L \text{trace } A(x) \, dx \leq \beta_1, \tag{2.21}$$

then there exists no nontrivial solution of (1.12). In fact, taking into account that for each $x \in [0, L]$, $\rho(x)$ is an eigenvalue of the matrix $A(x)$ and that in this case all the eigenvalues of $A(x)$, $\lambda_1(x), \cdots, \lambda_n(x)$, are nonnegative, we have $\rho(x) \leq \sum_{i=1}^n \lambda_i(x) = \text{trace } A(x)$ (see [15] for this last relation). Therefore, from (2.21) we obtain

$$\|\rho^+\|_1 = \int_0^L \rho(x) \, dx \leq \beta_1. \tag{2.22}$$

Previous remark shows that, if we want to have a criterion implying that (1.12) has only the trivial solution, then (2.22) is better than (2.21).

As in the scalar case, it may be seen that for Dirichlet boundary conditions, hypothesis [H] is not necessary. However, for Neumann boundary conditions, a restriction like [H] is natural (see Remark 4 and Remark 5 in [4]).

In Corollary 3.3 of the next section it is shown how, for elliptic systems, we can obtain optimal conditions without the help of the spectral radius of the matrix $A(x)$. Obviously that Corollary is also applicable to ordinary problems as (1.12).

3. Elliptic systems

This section will be concerned with linear boundary value problems of the form

$$\Delta u(x) + A(x)u(x) = 0,\ x \in \Omega,\quad \frac{\partial u(x)}{\partial n} = 0,\ x \in \partial\Omega, \tag{3.1}$$

Here $\Omega \subset \mathbb{R}^N,\ N \geq 2$ is a bounded and regular domain, $\dfrac{\partial}{\partial n}$ is the outer normal derivative on $\partial\Omega$ and $A \in \Lambda_*$, where Λ_* is defined as

[Λ_*] *The set of real $n \times n$ symmetric matrix valued function $A(\cdot)$, with continuous element functions $a_{ij}(x),\ 1 \leq i, j \leq n,\ x \in \overline{\Omega}$, such that (3.1) has not nontrivial constant solutions and*

$$\int_\Omega < A(x)k, k > dx \geq 0,\ \forall\ k \in \mathbb{R}^n. \tag{3.2}$$

In (3.1), $u \in (H^1(\Omega))^n$, the usual Sobolev space.

As in the ordinary case, we now collect some results which have been proved in [5].

Lemma 3.1. *([5]) If $1 \le \frac{N}{2} < p \le \infty$ is a given number, let us define the set X_p and the functional I_p as*

$$X_p = \left\{ v \in H^1(\Omega) : \int_\Omega |v|^{\frac{2}{p-1}} v = 0 \right\}, \quad \text{if } \frac{N}{2} < p < \infty,$$

$$I_p : X_p \setminus \{0\} \to \mathbb{R}, \; I_p(v) = \frac{\displaystyle\int_\Omega |\nabla v|^2}{\left(\displaystyle\int_\Omega |v|^{\frac{2p}{p-1}} \right)^{\frac{p-1}{p}}}, \; \text{if } \frac{N}{2} < p < \infty,$$

$$X_\infty = \{ v \in H^1(\Omega) : \int_\Omega v = 0 \}, \tag{3.3}$$

$$I_\infty : X_\infty \setminus \{0\} \to \mathbb{R}, \; I_\infty(v) = \frac{\displaystyle\int_\Omega |\nabla v|^2}{\displaystyle\int_\Omega v^2}$$

If

$$\beta_p \equiv \min_{X_p \setminus \{0\}} \; I_p, \; \frac{N}{2} < p \le \infty, \tag{3.4}$$

and a given function a satisfies

$$a \in L^p(\Omega, \mathbb{R}) \setminus \{0\}, \; \int_\Omega a \ge 0, \; \|a^+\|_p < \beta_p, \tag{3.5}$$

then the scalar problem

$$\Delta u(x) + a(x) u(x) = 0, \; x \in \Omega, \quad \frac{\partial u(x)}{\partial n} = 0, \; x \in \partial\Omega, \tag{3.6}$$

has only the trivial solution.

Remark 6. As in the ordinary case, $\beta_\infty = \lambda_1$, the first strictly positive eigenvalue of the Neumann eigenvalue problem in the domain Ω. Consequently, it seems difficult to obtain explicit expressions for β_p, as a function of p, Ω and N, at least for general domains. Finally, the constant β_p is optimal in the following sense: if $\frac{N}{2} < p \le \infty$ and

$$\Sigma_p^* = \{ a \in L^p(\Omega) \setminus \{0\} : \int_\Omega a(x)\, dx \ge 0 \text{ and (3.1) has nontrivial solutions } \}$$

then

$$\beta_p \equiv \min_{a \in \Sigma_p^*} \; \|a^+\|_p, \; N/2 < p \le \infty.$$

Next result may be proved by using the same ideas as in Theorem 2.3.

Theorem 3.2. *Let $A(\cdot) \in \Lambda_*$ be such that there exist a diagonal matrix $B(x)$ with continuous entries $b_{ii}(x)$, and numbers $p_i \in (N/2, \infty],\ 1 \le i \le n$, which fulfil*

$$\begin{aligned} &A(x) \le B(x),\ \forall\ x \in \overline{\Omega} \\ &\|b_{ii}^{+}\|_{p_i} < \beta_{p_i},\ 1 \le i \le n. \end{aligned} \tag{3.7}$$

Then, there exists no nontrivial solution of the vector boundary value problem (3.1).

Remark 7. As in the ordinary case, the previous Theorem is optimal in the sense of Remark 3 (see Theorem 2.1 in [5]). Moreover, by using the previous Theorem, it is possible to obtain a corollary similar to corollary 2.4, which involves the spectral radius $\rho(x)$ of the matrix $A(x)$ and the norm $\|\rho^{+}\|_{p}$. The unique difference with the ordinary case is that, for elliptic systems, $p \in (N/2, \infty]$.

In the next Corollary and in order to show how our Theorem 3.2 can be used without the help of the spectral radius of the matrix $A(x)$, we consider the case of a system with two equations.

Corollary 3.3. *Let the matrix $A(x)$ be given by*

$$A(x) = \begin{pmatrix} a_{11}(x) & a_{12}(x) \\ a_{12}(x) & a_{22}(x) \end{pmatrix} \tag{3.8}$$

where

[H1]
$$\begin{aligned} &a_{ij} \in C(\overline{\Omega}),\ 1 \le i, j \le 2, \\ &a_{11}(x) \ge 0,\ a_{22}(x) \ge 0,\ a_{11}(x)a_{22}(x) \ge a_{12}^2(x),\ \forall\ x \in \overline{\Omega}, \\ &\det\ A(x) \ne 0,\ \text{for some } x \in \overline{\Omega}. \end{aligned}$$

In addition, let us assume that there exist $p_1, p_2 \in (N/2, \infty]$ such that

$$\|a_{11}\|_{p_1} < \beta_{p_1},\quad \|a_{22} + \frac{a_{12}^2}{\beta_{p_1} - \|a_{11}\|_{p_1}}\|_{p_2} < \beta_{p_2}. \tag{3.9}$$

Then the unique solution of (3.1) is the trivial one.

Proof. It is trivial to see that [**H1**] implies that the eigenvalues of the matrix $A(x)$ are both nonnegative, which implies that $A(x)$ is positive semi-definite. Also, since $\det\ A(x) \ne 0,$ for some $x \in \overline{\Omega}$, (3.1) has not nontrivial constant solutions. Therefore, $A(\cdot) \in \Lambda_*$. Moreover, it is easy to check that for a given diagonal matrix $B(x)$, with continuous entries $b_{ii}(x),\ 1 \le i \le 2$, the relation

$$A(x) \le B(x),\ \forall\ x \in \overline{\Omega} \tag{3.10}$$

is satisfied if and only if $\forall\ x \in \overline{\Omega}$, we have

$$\begin{array}{c} b_{11}(x) \geq a_{11}(x),\ b_{22}(x) \geq a_{22}(x), \\ (b_{11}(x) - a_{11}(x))(b_{22}(x) - a_{22}(x)) \geq a_{12}^2(x). \end{array} \tag{3.11}$$

In our case, if we choose

$$b_{11}(x) = a_{11}(x) + \gamma,\ b_{22}(x) = a_{22}(x) + \frac{a_{12}^2(x)}{\gamma} \tag{3.12}$$

where γ is any constant such that

$$\begin{array}{c} 0 < \gamma < \beta_{p_1} - \|a_{11}\|_{p_1}, \\ \left(\frac{1}{\gamma} - \frac{1}{\beta_{p_1} - \|a_{11}\|_{p_1}}\right) \|a_{12}^2\|_{p_2} < \beta_{p_2} - \|a_{22} + \dfrac{a_{12}^2}{\beta_{p_1} - \|a_{11}\|_{p_1}}\|_{p_2} \end{array} \tag{3.13}$$

then all conditions of Theorem 3.2 are fulfilled and consequently (3.1) has only the trivial solution. □

Remark 8. Previous corollary may be seen as a perturbation result in the following sense: let us assume that we have an uncoupled system of the type

$$\begin{array}{c} \Delta u_1(x) + a_{11}(x) u_1(x) = 0,\ x \in \Omega;\ \ \frac{\partial u_1(x)}{\partial n} = 0\ \ x \in \partial\Omega, \\ \Delta u_2(x) + a_{22}(x) u_2(x) = 0,\ x \in \Omega;\ \ \frac{\partial u_2(x)}{\partial n} = 0\ \ x \in \partial\Omega, \end{array} \tag{3.14}$$

where

$$\begin{array}{c} a_{ii} \in C(\overline{\Omega}),\ 1 \leq i \leq 2,\ a_{11}(x) \geq \delta > 0,\ a_{22}(x) \geq \delta,\ \forall\ x \in \overline{\Omega}. \\ \exists\ p_1, p_2 \in (N/2, \infty] :\ \|a_{11}\|_{p_1} < \beta_{p_1},\ \ \|a_{22}\|_{p_2} < \beta_{p_2}. \end{array} \tag{3.15}$$

Then it is clear from the scalar results (see Remark 6) that the unique solution of (3.14) is the trivial one (see Corollary 6.1 in [5]). Now, we can use Corollary 3.3 to ensure the permanence of the uniqueness property (with respect to the existence of solutions) of the coupled system (3.1), for any function $a_{12} \in C(\overline{\Omega})$ with $L^\infty-$norm sufficiently small. Here we have considered that the functions $a_{ii}(x),\ 1 \leq i \leq 2$, are fixed and that the uncoupled system is perturbed by the function $a_{12}(x)$. But it is clear that we may consider, for example, $a_{11}(x), a_{12}(x)$ fixed and $a_{22}(x)$ as the perturbation. Some of these results may be generalized to systems with n equations. For example, if we have an uncoupled system of the type

$$\Delta u_i(x) + a_{ii}(x) u_i(x) = 0,\ x \in \Omega;\ \ \frac{\partial u_i(x)}{\partial n} = 0\ \ x \in \partial\Omega,\ 1 \leq i \leq n \tag{3.16}$$

where

$$\begin{array}{c} a_{ii} \in C(\overline{\Omega}),\ 1 \leq i \leq n,\ a_{ii}(x) \geq \delta > 0,\ 1 \leq i \leq n,\ \forall\ x \in \overline{\Omega}, \\ \exists\ p_i, \in (N/2, \infty] :\ \|a_{ii}\|_{p_i} < \beta_{p_i},\ 1 \leq i \leq n, \end{array} \tag{3.17}$$

then we can use Theorem 3.2 to ensure the permanence of the uniqueness property (with respect to the existence of solutions) of the coupled system (3.1), for any functions $a_{ij} = a_{ji} \in C(\overline{\Omega}),\ 1 \leq i \neq j \leq n$ with $L^\infty-$norm

sufficiently small. The proof is similar to the case of two equations and it is based on Theorem 3.2. The unique difference is that now, the matrix $B(x)$ is given by $b_{ii}(x) = a_{ii}(x)+\varepsilon,\ 1 \le i \le n$ with ε sufficiently small. It is easily deduced that if the $L^\infty-$norm of the functions $a_{ij} = a_{ji},\ 1 \le i \ne j \le n$ are sufficiently small, then the matrix $B(x) - A(x)$ is positive definite for all $x \in \overline{\Omega}$.

Next we give some new results on the existence and uniqueness of solutions of nonlinear resonant problems. We prefer to deal with systems of P.D.E. (similar results can be proved for ordinary differential systems; in this last case it is possible to choose the constants $p_i \in [1,\infty],\ 1 \le i \le n$). In particular, next Theorem is a generalization, for systems of equations, of the main result given in [22] for the Neumann problem. Moreover, it is a generalization (at the two first eigenvalues of (1.5)) of some results given in [2] and [13] where the authors take all the constants $p_i = \infty,\ 1 \le i \le n$.

In the proof, the basic idea is to combine the results obtained in the linear case with Schauder's fixed point theorem.

Theorem 3.4. *Let $\Omega \subset \mathbb{R}^N$ $(N \ge 2)$ be a bounded and regular domain and $G : \overline{\Omega} \times \mathbb{R}^n \to \mathbb{R},\ (x,u) \to G(x,u)$ satisfying:*

(1) (a) *$u \to G(x,u)$ is of class $C^2(\mathbb{R}^n,\mathbb{R})$ for every $x \in \overline{\Omega}$.*
    (b) *$x \to G(x,u)$ is continuous on $\overline{\Omega}$ for every $u \in \mathbb{R}^n$.*

(2) *There exist continuous matrix functions $A(\cdot)$, $B(\cdot)$, with $B(x)$ diagonal and with entries $b_{ii}(x)$, and $p_i \in (N/2,\infty]$ $1 \le i \le n$, such that*

$$\left.\begin{array}{c} A(x) \le G_{uu}(x,u) \le B(x) \ \text{in}\ \overline{\Omega} \times \mathbb{R}^n, \\[2ex] \|b_{ii}^+\|_{p_i} < \beta_{p_i},\ 1 \le i \le n, \\[2ex] \int_\Omega < A(x)k, k > dx > 0,\ \forall\ k \in \mathbb{R}^n \setminus \{0\}. \end{array}\right\} \tag{3.18}$$

Then system

$$\left.\begin{array}{c} \Delta u(x) + G_u(x,u(x)) = 0,\ x \in \Omega, \\ \frac{\partial u(x)}{\partial n} = 0, \quad x \in \partial\Omega, \end{array}\right\} \tag{3.19}$$

has a unique solution.

Proof. We first prove uniqueness. Let v and w be two solutions of (3.19). Then, the function $u = v - w$ is a solution of the problem

$$\Delta u(x) + C(x)u(x) = 0,\ x \in \Omega,\ \frac{\partial u}{\partial n} = 0,\ x \in \partial\Omega \tag{3.20}$$

where $C(x) = \displaystyle\int_0^1 G_{uu}(x, w(x) + \theta u(x))\ d\theta$ (see [16], p. 103, for the mean value theorem for the vectorial function $G_u(x,u)$). Hence $A(x) \le C(x) \le B(x)$ and we deduce that $C(x)$ satisfies all the hypotheses of Theorem 3.2.

Consequently, $u \equiv 0$.
Next we prove existence. First, we write (3.19) in the equivalent form

$$\left.\begin{array}{lr} \Delta u(x) + D(x,u(x))u(x) + G_u(x,0) = 0, & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega \end{array}\right\} \tag{3.21}$$

where the function $D : \overline{\Omega}\times\mathbb{R}^n \to \mathcal{M}(\mathbb{R})$ is defined by $D(x,z) = \displaystyle\int_0^1 G_{uu}(x,\theta z)\,d\theta$. Here $\mathcal{M}(\mathbb{R})$ denotes the set of real $n \times n$ matrices. Let $X = (C(\overline{\Omega}))^n$ be with the uniform norm, i.e., if $y(\cdot) = (y^1(\cdot), \cdots, y^n(\cdot)) \in X$, then $\|y\|_X = \displaystyle\sum_{k=1}^n \|y^k(\cdot)\|_\infty$. Since

$$A(x) \leq D(x,z) \leq B(x), \ \forall\ (x,z) \in \overline{\Omega} \times \mathbb{R}^n, \tag{3.22}$$

we can apply Theorem 3.2 in order to have a well defined operator $T : X \to X$, by $Ty = u_y$, being u_y the unique solution of the linear problem

$$\left.\begin{array}{lr} \Delta u(x) + D(x,y(x))u(x) + G_u(x,0) = 0, & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega. \end{array}\right\} \tag{3.23}$$

We will show that T is completely continuous and that $T(X)$ is bounded. The Schauder's fixed point theorem provides a fixed point for T which is a solution of (3.19).

The fact that T is completely continuous is a consequence of the compact embedding of the Sobolev space $W^{2,q}(\Omega) \subset C(\overline{\Omega})$ for q sufficiently large. It remains to prove that $T(X)$ is bounded. Suppose, contrary to our claim, that $T(X)$ is not bounded. In this case, there would exist a sequence $\{y_n\} \subset X$ such that $\|u_{y_n}\|_X \to \infty$. From (3.22), and passing to a subsequence if necessary, we may assume that, for each $1 \leq i,j \leq n$, the sequence of functions $\{D_{ij}(\cdot, y_n(\cdot))\}$ is weakly convergent in $L^p(\Omega)$ to a function $E_{ij}(\cdot)$ and such that if $E(x) = (E_{ij}(x))$, then $A(x) \leq E(x) \leq B(x)$, a.e. in Ω, ([18], page 157).

If $z_n \equiv \dfrac{u_{y_n}}{\|u_{y_n}\|_X}$, passing to a subsequence if necessary, we may assume that $z_n \to z_0$ strongly in X (we have used again the compact embedding $W^{2,q}(\Omega) \subset C(\overline{\Omega})$), where z_0 is a nonzero vectorial function satisfying

$$\left.\begin{array}{lr} \Delta z_0(x) + E(x)z_0(x) = 0, & \text{in } \Omega, \\ \frac{\partial z_0}{\partial n} = 0, & \text{on } \partial\Omega \end{array}\right\} \tag{3.24}$$

This is a contradiction with Theorem 3.2. □

References

[1] S. Ahmad, *An existence theorem for periodically perturbed conservative systems,* Michigan Math. J., 20, (1973), 385-392.

[2] P.W. Bates, *Solutions of nonlinear elliptic systems with meshed spectra,* Nonlinear Anal., 4 (1979), 1023-1030.

[3] K.J. Brown and S.S. Lin, *Periodically perturbed conservative systems and a global inverse function theorem,* Nonlinear Anal., 4, (1980), 193-201.

[4] A. Cañada, J.A. Montero and S. Villegas, *Liapunov-type inequalities and Neumann boundary value problems at resonance*, MIA, Mathematical Inequalities and Applications, 8 (2005), 459-475.

[5] A. Cañada, J.A. Montero and S. Villegas, *Lyapunov inequalities for partial differential equations*, J. Funct. Anal., 237, (2006), 176-193.

[6] A. Cañada and S. Villegas, *Optimal Lyapunov inequalities for disfocality and Neumann boundary conditions using L^p norms,* Discrete and Continuous Dynamical Systems, 20, (2008), 877-888.

[7] A. Cañada and S. Villegas, *Lyapunov inequalities for Neumann Boundary Conditions at Higher Eigenvalues,* to appear in Journal of the European Mathematical Society.

[8] S. Clark and D. Hinto, *A Liapunov inequality for linear hamiltonian systems,* MIA, Mathematical Inequalities and Applications, 1, (1998), 201-209.

[9] S. Clark and D. Hinton, *Positive eigenvalues of second order boundary value problems and a theorem of M.G. Krein,* Proc. Amer. Math. Soc., 130, (2002), 3005-3015.

[10] A. Fonda and J. Mawhin, Iterative and variational methods for the solvability of some semilinear equations in Hilbert spaces, J. Diff. Equations, 98, (1992), 355-375.

[11] P. Hartman, *Ordinary Differential Equations,* John Wiley and Sons Inc., New York-London-Sydney, 1964.

[12] W. Huaizhong and L. Yong, *Neumann boundary value problems for second-order ordinary differential equations across resonance,* SIAM J. Control and Optimization, 33, (1995), 1312-1325.

[13] J. L. Kazdan and F. W. Warner, *Remarks on some queasilinear elliptic equations,* Comm. Pure Appl. Math. 28, (1975), 567-597.

[14] A.C. Lazer, *Application of a lemma on bilinear forms to a problem in nonlinear oscillations,* Proc. Amer. Math. Soc., 33, (1972), 89-94.

[15] S. Lang, *Algebra,* Addison-Wesley Publishing Company, INC., 1965.

[16] S. Lang, *Analysis.II,* Addison-Wesley, Reading, Mass., 1969.

[17] A.C. Lazer and D.A. Sánchez, *On periodically perturbed conservative systems,* Mich. Math. J, 16, (1969), 193-200.

[18] E.B. Lee and L. Markus, *Foundations of Optimal Control Theory,* Wiley, New York, 1967.

[19] J. Mawhin, *Contractive mappings and periodically perturbed conservative systems,* Arch. Math. (Brno), 12, (1976), 67-74.

[20] J. Mawhin, *Conservative systems of semilinear wave equations with periodic-Dirichlet boundary conditions,* J. Differential Equations, 42, (1981), 116-128.

[21] J.Mawhin and J.R. Ward, *Nonuniform nonresonance conditions at the two first eigenvalues for periodic solutions of forced Liénard and Duffing equations*, Rocky Mountain J. Math., 12, (1982), 643-654.

[22] J. Mawhin, J.R. Ward and M. Willem, *Variational methods and semilinear elliptic equations,* Arch. Rational Mech. Anal., 95, (1986), 269-277.

[23] W. T. Reid, *A matrix Liapunov inequality,* J. Math. Anal. Appl., 32, (1970), 424-434.

[24] W. T. Reid, *A generalizede Liapunov inequality,* J. Diff. Eqns., 13, (1973), 182-196.

[25] W. T. Reid, *Interrelations between a trace formula and Liapunov type inequalities,* J. Diff. Eqns., 23, (1977), 448-458.

[26] J.R. Ward, *The existence of periodic solutions for nonlinear perturbed conservative systems,* Nonlinear Anal., 3, (1979), 697-705.

[27] M. Zhang, *Certain classes of potentials for p-Laplacian to be non-degenerate*, Math. Nachr., 278, (2005), 1823-1836.

Departamento de Análisis Matemático, Universidad de Granada, 18071 Granada, Spain.

E-mail address: `acanada@ugr.es, svillega@ugr.es`